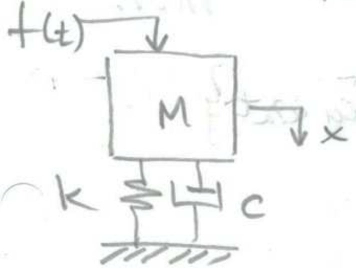
VIBRACIONES TRANSITORIAS:

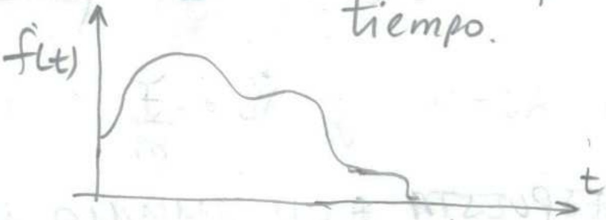
RESPUESTA A EXCITACIONES NO PERIÓDICAS

Para resolver un sistema excitado con cualquier fuerza externa se pueden utilizar varios métodos:

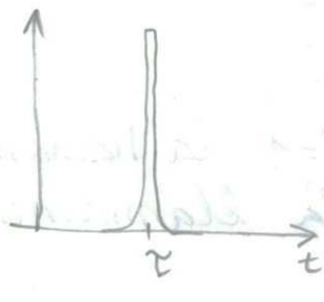
1.- INTEGRAL DE CONVOLUCIÓN
2.- INTEGRAL DE DUHAMEL
3.- TRANSFORMADA DE LAPLACE
4.- SUPERPOSICIÓN
5.- REPRESENTACIÓN DE LA EXCITACIÓN POR TRANSFORMADA DE FOURIER.



$f(t)$ = cualquier función que cesa en el tiempo.



EJ.1 - $f(t)$ = FUERZA IMPULSIVA.



$$\hat{F} = \int_0^\infty f(t)\,dt = \int_{\tau-\varepsilon}^{\tau+\varepsilon} f(t)\,dt$$

$$f(t) = F_0\,\delta(t-\tau) = \begin{cases} 0 & t \neq \tau \\ F_0 & t = \tau \end{cases}$$

Impulso unitario
se denomina también Delta Dirac

Si tenemos un impulso en $t=0$

¿cual es la respuesta en el tiempo?

La ecuación: $m\ddot{x}+c\dot{x}+kx=f(t)$

Conocemos que cuando se aplica un impulso: $\bar{f}=\frac{d\bar{p}}{dt}$

$$\hat{\mathcal{F}}=\int_0^{\infty}f(t)\,dt=\int m\frac{dv}{dt}dt=m\Delta v$$ se transforma en un cambio de velocidad.

Por tanto $x(t=0)=x_0=0$

$\dot{x}(t=0)=\dot{x}_0 \longrightarrow \mathcal{F}=mv_f-m\cancel{v_0}^{0}$

$\boxed{v_f=\frac{\mathcal{F}}{m}}$ ← representa la velocidad inicial del mov.

Recordamos la respuesta a cond. iniciales:

$$x(t)=e^{-\xi\omega_n t}\left\{x_0\cos\omega_a t+\left(\frac{\dot{x}_0+\xi\omega_n x_0}{\omega_a}\right)\operatorname{sen}\omega_a t\right\}$$

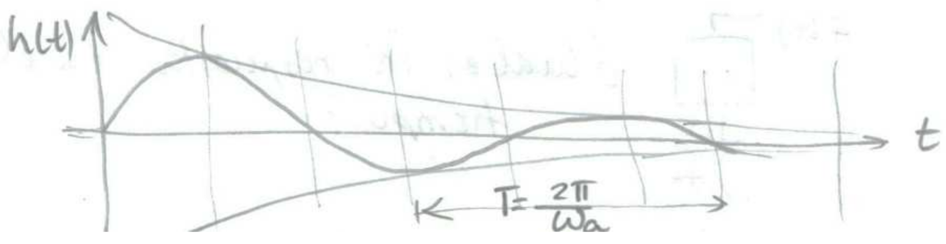
Como $x_0=0$ y $\dot{x}_0=\frac{\mathcal{F}}{m}$

LA RESPUESTA A UN IMPULSO SERA:

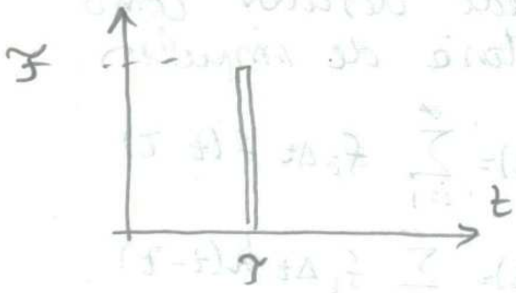
$$\boxed{x(t)=\frac{\mathcal{F}}{m\omega_a}e^{-\xi\omega_n t}\operatorname{sen}\omega_a t}$$

Si el impulso es unitario $\mathcal{F}=1$ la llamamos la respuesta impulsiva y la llamamos $h(t)$

$$\boxed{h(t)=\frac{e^{-\xi\omega_n t}}{m\omega_a}\operatorname{sen}\omega_a t}$$

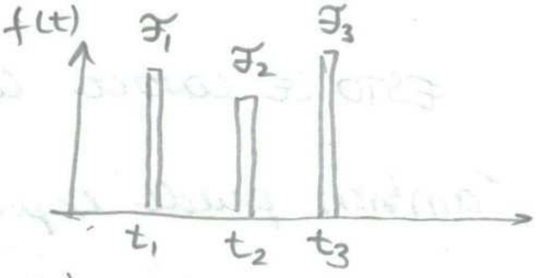


Si el impulso $\mathcal{F}$ se aplica en un tiempo cualquiera τ, la respuesta será la misma pero desfasada τ.



y se puede expresar como:

$$\underline{x(t) = \mathcal{F}\, h(t-\tau)}$$



Si se aplican varios impactos

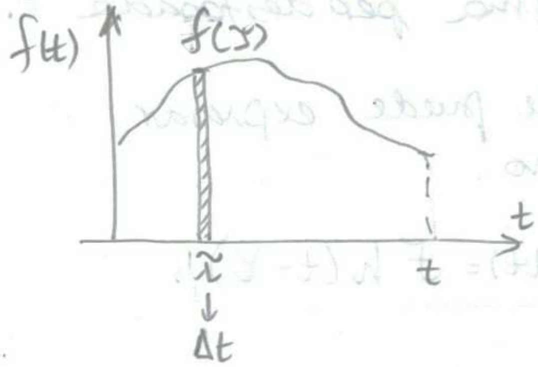
$$f(t) = \mathcal{F}_1 \delta(t-t_1) + \mathcal{F}_2 \delta(t-t_2) + \mathcal{F}_3 \delta(t-t_3)$$

La respuesta sera por partes dependiendo en que tiempo la estoy evaluando la respuesta:

	Para
$x(t) = 0$	$0 < t < t_1$
$x(t) = \mathcal{F}_1 h(t-t_1)$	$t_1 < t < t_2$
$x(t) = \mathcal{F}_1 h(t-t_1) + \mathcal{F}_2 h(t-t_2)$	$t_2 < t < t_3$
$x(t) = \mathcal{F}_1 h(t-t_1) + \mathcal{F}_2 h(t-t_2) + \mathcal{F}_3 h(t-t_3)$	$t > t_3$

Con $h(t-t_i) = \dfrac{1}{w_a m} e^{-\xi(t-t_i)} \operatorname{sen} w_a (t-t_i)$

Respuesta a una fuerza general



Se puede describir como sumatoria de impulsos:

$$f(t)=\sum_{i=1}^{\infty} f_i \Delta t \,\delta(t-\tau)$$

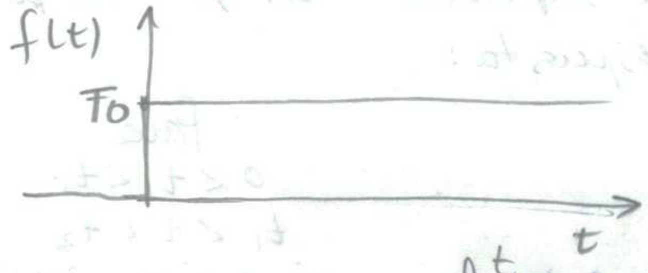
$$x(t)=\sum_{i=1}^{\infty} f_i \Delta t \,h(t-\tau)$$

para $\lim_{i\to\infty} \Delta t \to dt \to \boxed{x(t)=\int_0^t f(\tau)\,h(t-\tau)\,d\tau}$

ESTO SE CONOCE COMO INTEGRAL DE CONVOLUCIÓN

También puede expresarse como $\boxed{x(t)=\int_0^t f(t-\tau)\,h(\tau)\,d\tau}$

RESPUESTA A UN ESCALÓN



$f(t)=F_0 \qquad \tau=0$

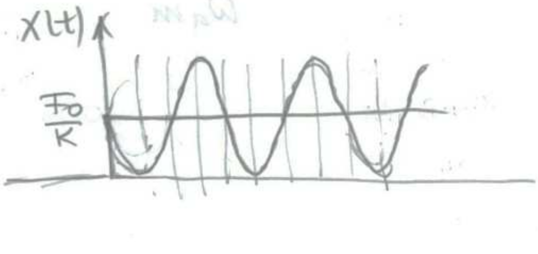
$$x(t)=\int_0^t F_0 \frac{e^{-\xi\omega_n\tau}}{m\omega_a}\,\mathrm{sen}\,\omega_a\tau\,d\tau$$

Recordar: $\int u\,dv = uv-\int v\,du$

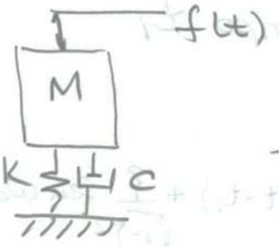
Si $\xi=0 \quad x(t)=\int_0^t \frac{F_0}{m\omega_n}\,\mathrm{sen}\,\omega_n\tau\,d\tau = -\frac{F_0}{m\omega_n}\frac{\cos\omega_n\tau}{\omega_n}\Big|_0^t$

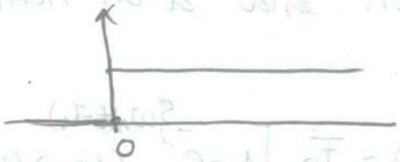
$$x(t)=-\frac{F_0}{m\omega_n^2}\left(\cos\omega_n t-\cos 0\right)$$

$\therefore \boxed{x(t)=\frac{F_0}{K}\left(1-\cos\omega_n t\right)}$



RESPUESTA A UN ESCALÓN: SISTEMA AMORTIGUADO





$$f(t) = F_0$$
$$h(t) = \frac{e^{-\xi\omega_n t}}{m\omega_a}\,\mathrm{sen}\,\omega_a t$$

$$x(t) = \int f(\tau)\,h(t-\tau)\,d\tau = \int f(t-\tau)\,h(\tau)\,d\tau$$

Recordar $\int u\,dv = uv - \int v\,du$

$u = \mathrm{sen}\,\omega_a\tau \quad du = \omega_a \cos\omega_a\tau$
$dv = e^{-\xi\omega_n\tau} \quad v = -\frac{e^{-\xi\omega_n\tau}}{\xi\omega_n}$

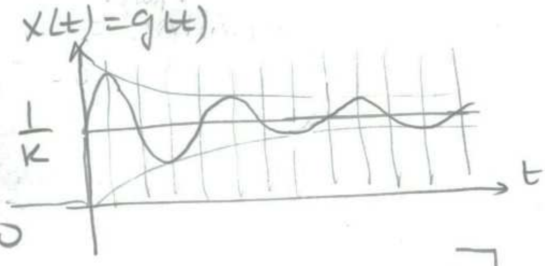
$$x(t) = \int \frac{F_0}{m\omega_a} e^{-\xi\omega_n(t)}\,\mathrm{sen}\,\omega_a\tau\,d\tau = \frac{F_0}{m\omega_a}\left[-\frac{e^{-\xi\omega_n t}}{\xi\omega_n}\,\mathrm{sen}\,\omega_a\tau\Big|_0^t - \int -\frac{e^{-\xi\omega_n t}}{\xi\omega_n}\,\omega_a\cos\omega_a\tau\,d\tau\right.$$

Vuelvo a aplicar $u = \cos\omega_a t \quad du = -\omega_a\,\mathrm{sen}\,\omega_a\tau$
$dv = e^{-\xi\omega_n\tau} \quad v = -\frac{e^{-\xi\omega_n t}}{\xi\omega_n}$

$$= \frac{F_0}{m\omega_a}\left[-\frac{e^{-\xi\omega_n t}}{\xi\omega_n}\,\mathrm{sen}\,\omega_a t + \frac{\omega_a}{\xi\omega_n}\left[\frac{e^{-\xi\omega_n\tau}}{\xi\omega_n}\cos\omega_a\tau\Big|_0^t - \int_0^t -\frac{e^{-\xi\omega_n\tau}}{\xi\omega_n}(-\omega_a)\,\mathrm{sen}\,\omega_a\tau\,d\tau\right]\right]$$

$$= \frac{F_0}{m\omega_a}\left[-\frac{e^{-\xi\omega_n t}}{\xi\omega_n}\,\mathrm{sen}\,\omega_a t + \frac{\omega_a}{\xi^2\omega_n^2}\left[\left(e^{-\xi\omega_n t}\cos\omega_a t\right) - 1\right] - \frac{\omega_a^2}{\xi^2\omega_n^2}\underbrace{\int e^{-\xi\omega_n\tau}\,\mathrm{sen}\,\omega_a\tau\,d\tau}_{\text{misma integral}}\right.$$

$$\therefore \int e^{-\xi\omega_n\tau}\,\mathrm{sen}\,\omega_a\tau\,d\tau\underbrace{\left[1 + \frac{1-\xi^2}{\xi^2}\right]}_{\text{“}\frac{1}{\xi^2}\text{”}} = -\frac{e^{-\xi\omega_n t}}{\xi\omega_n}\,\mathrm{sen}\,\omega_a t + \frac{\omega_a}{\xi\omega_n^2}\left[e^{-\xi\omega_n t}\cos\omega_a t - 1\right]$$

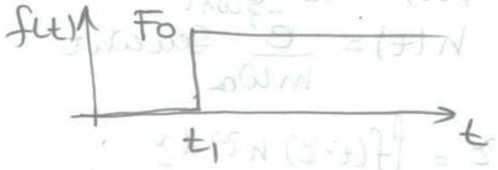


⋮

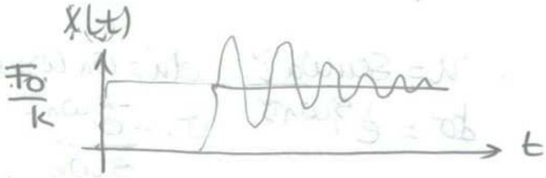
RESPUESTA AL ESCALÓN UNITARIO

$$x(t) = g(t) = \frac{1}{K}\left[1 - e^{-\xi\omega_n t}\left(\cos\omega_a t + \frac{\xi}{\sqrt{1-\xi^2}}\,\mathrm{sen}\,\omega_a t\right)\right]$$

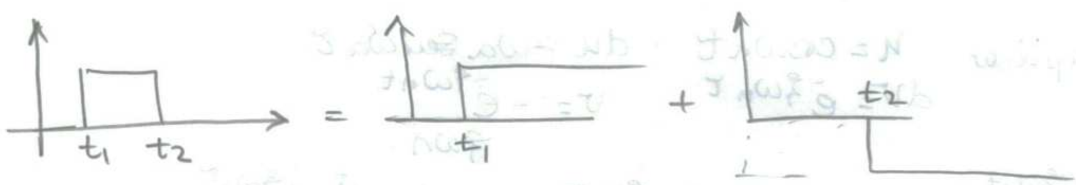
Si tenemos un escalón esta a un tiempo t_1



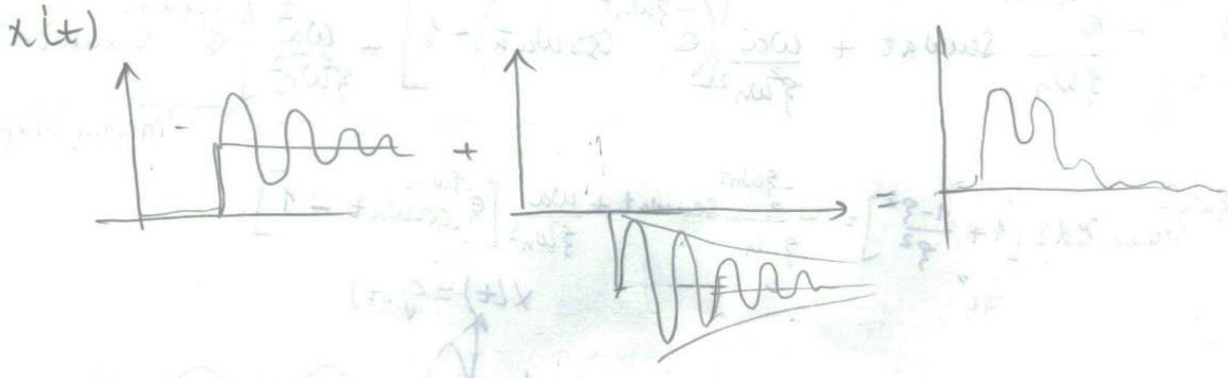
$$x(t) = \frac{F_0}{K}\left[1 - e^{-\xi\omega_n(t-t_1)}\left(\cos\omega_d(t-t_1) + \frac{\xi}{\sqrt{1-\xi^2}}\,\operatorname{sen}\omega_d(t-t_1)\right)\right]$$



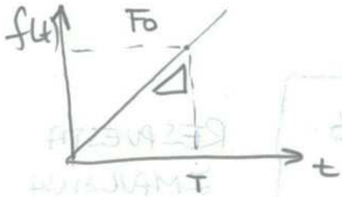
Si tenemos una excitacion compuesta por 2 escalones



La respuesta será: $x(t) = F_0\, g(t-t_1) - F_0\, g(t-t_2)$



RESPUESTA A UNA RAMPA



$$f(t) = \frac{F_0}{T} t \qquad x(t) = \int_0^t f(\tau)\, h(t-\tau)\, d\tau = \int_0^t f(t-\tau)\, h(\tau)\, d\tau$$

para un sistema no amortiguado $h(\tau) = \frac{1}{m\omega_n} \operatorname{sen} \omega_n \tau$

$$x(t) = \int_0^t \frac{F_0}{T}(t-\tau) \cdot \frac{1}{m\omega_n} \operatorname{sen} \omega_n \tau \, d\tau$$

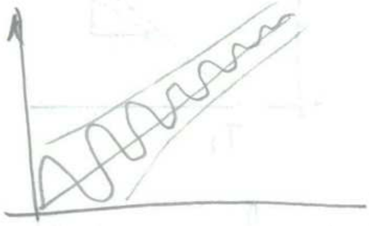
$$u = \frac{F_0}{T}(t-\tau) \qquad dv = \operatorname{sen} \omega_n \tau \qquad \int u\,dv = uv - \int v\,du$$

$$du = -\frac{F_0}{T} \qquad v = -\frac{\cos \omega_n \tau}{\omega_n}$$

$$x(t) = \frac{1}{m\omega_n}\left[\frac{F_0}{T}(t-\tau)\left(-\frac{\cos\omega_n \tau}{\omega_n}\right)\Big|_0^t - \int_0^t -\frac{\cos \omega_n \tau}{\omega_n}\left(-\frac{F_0}{T}\right) d\tau \right]$$

$$x(t) = \frac{1}{m\omega_n}\left[\frac{F_0\, t}{T\omega_n} - \frac{F_0}{T\omega_n} \frac{\operatorname{sen} \omega_n \tau}{\omega_n}\Big|_0^t \right] = \frac{F_0}{T m \omega_n^2}\left(t - \frac{1}{\omega_n} \operatorname{sen} \omega_n t\right)$$

$$\boxed{x(t) = r(t) = \frac{F_0}{KT}\left(t - \frac{1}{\omega_n} \operatorname{sen} \omega_n t\right)}$$



para sist. amortiguado

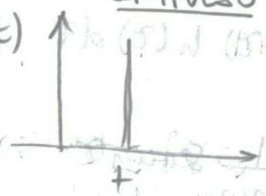
$$\boxed{r(t) = \frac{1}{k}\left[t - \frac{2\xi}{\omega_n}\left(1 - e^{-\xi\omega_n t} \cos \omega_a t\right) + \frac{2\xi^2 - 1}{\omega_a} e^{-\xi \omega_n t} \operatorname{sen} \omega_a t \right]}$$

RESUMEN

EXCITACIÓN $f(t)$	RESPUESTA $x(t)$

IMPULSO

$\delta(t)$



$$\boxed{h(t) = \frac{1}{m\omega_a} e^{-\xi\omega_n t} \operatorname{sen}\omega_a t}$$

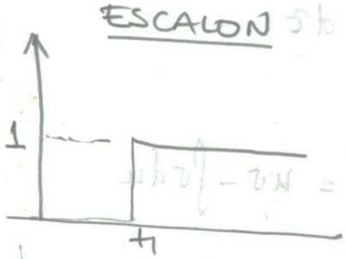
RESPUESTA IMPULSIVA

INTEGRO

INTEGRO

ESCALÓN

$u(t)$



$$g(t) = \int_0^t h(\tau)\,d\tau$$

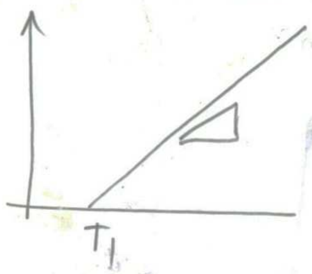
$$\boxed{g(t) = \frac{1}{k}\left[1 - e^{-\xi\omega_n t}\left(\cos\omega_a t + \frac{\xi}{\sqrt{1-\xi^2}}\operatorname{sen}\omega_a t\right)\right]}$$

RESPUESTA INDICIAL

$$u(t) = \int \delta(t - t_1)\,dt = 1$$

INTERGRO

INTEGRO



$$\boxed{r(t) = \frac{1}{k}\left[t - \frac{2\xi}{\omega_n}\left(1 - e^{-\xi\omega_n t}\cos\omega_a t\right) + \frac{2\xi^2 - 1}{\omega_a} e^{-\xi\omega_n t}\operatorname{sen}\omega_a t\right]}$$

$$r(t) = \int u(t - t_1)\,dt$$

$$= t(t - t_1)\,dt$$

RESPUESTA A UNA RAMPA.

Integral de Duhamel

Partimos de integral de convolución: $x(t) = \int_0^t f(\tau)\, h(t-\tau)\, d\tau$

y aplicamos integración por partes $\int u\,dv = uv\Big|_0^t - \int_0^t v\,du$

donde $u = f(\tau) \quad du = f'(\tau)$
$dv = h(t-\tau) \quad v = g(t-\tau)(-1)$

$$\therefore \; x(t) = -f(\tau).\,g(t-\tau)\Big|_0^t - \int_0^t f'(\tau)\, g(t-\tau)\, d\tau$$

$$= -f(t)\,g(0) + f(0)\,g(t) + \int_0^t f'(\tau)\, g(t-\tau)\, d\tau$$

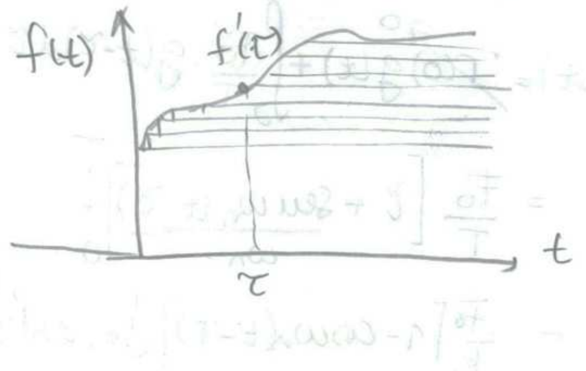
recordamos:

$$g(t) = \frac{1}{k}\left[1 - e^{\xi\omega_n t}\left(\cos\omega_a t + \frac{\xi}{\sqrt{1-\xi^2}}\operatorname{sen}\omega_a t\right)\right]$$

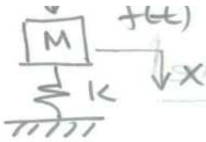
$$g(0) = \frac{1}{k}\left[1 - 1\right] = 0$$

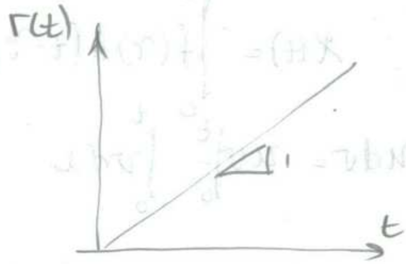
$$\therefore \; \boxed{x(t) = f(0)\,g(t) + \int_0^t f'(\tau)\, g(t-\tau)\, d\tau} \quad \text{INTEGRAL DE DUHAMEL}$$

Interpretación física:



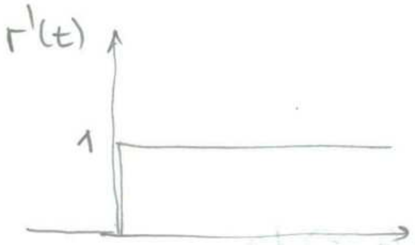
EJEMPLO





$$x(t) = \cancelto{0}{r(0)}\, g(t) + \int_0^t r'(0)\, g(t-\tau)\, d\tau$$

$$g(t) = \frac{1}{k}\left[1 - \cos\omega_n t\right]$$



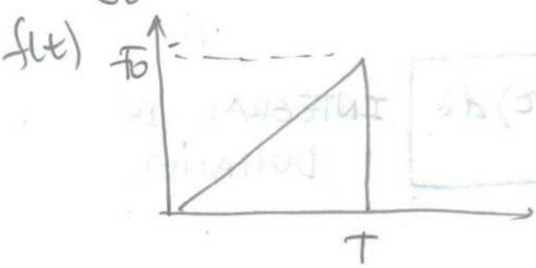
$$x(t) = \int_0^t \frac{1}{k}\left[1 - \cos\omega_n(t-\tau)\right] d\tau$$

$$= \frac{1}{k}\left[\tau + \frac{\operatorname{sen}\omega_n(t-\tau)}{\cancel{-}(\omega_n)}\right]\Bigg|_0^t$$

$$= \frac{1}{k}\left[t + \cancelto{0}{\frac{\operatorname{sen}\omega_n(0)}{\omega_n}} - \frac{\operatorname{sen}\omega_n t}{\omega_n}\right].$$

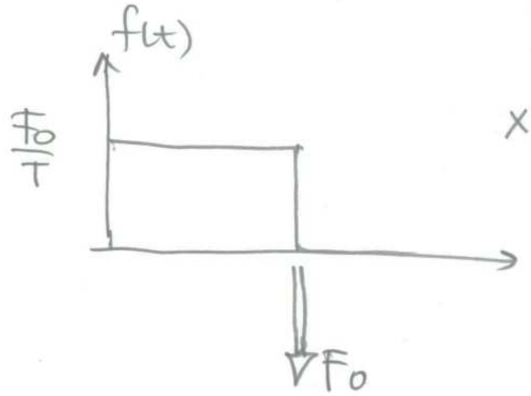
$$\boxed{x(t) = \frac{1}{k}\left[t - \frac{\operatorname{sen}\omega_n t}{\omega_n}\right]}$$

Si



Para $0 < t < T$

$$x(t) = \frac{F_0}{T k}\left[t - \frac{\operatorname{sen}\omega_n t}{\omega_n}\right.$$



Para $t > T$

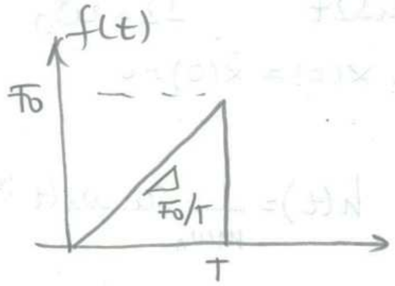
$$x(t) = \cancelto{0}{f(0)}\, g(t) + \int_0^T \frac{F_0}{T}\, g(t-\tau)\, d\tau + \int_{T-\varepsilon}^{T+\varepsilon} \underbrace{f(\tau)\, g(t-\tau)}_{cte}\, d\tau$$

$$= \frac{F_0}{T}\left[\tau + \frac{\operatorname{sen}\omega_n(t-\tau)}{\omega_n}\right]_0^T +$$

$$+ \frac{F_0}{k}\left[1 - \cos\omega_n(t-T)\right] \underbrace{\int \delta(\tau)\, d\tau}_{1}$$

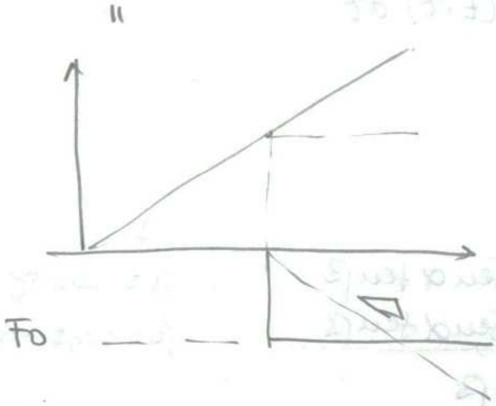
$$\therefore\ x(t) = \frac{F_0}{T}\left[T + \frac{\operatorname{sen}\omega_n(t-T)}{\omega_n} - \frac{\operatorname{sen}\omega_n t}{\omega_n}\right] - \frac{F_0}{k}\left[1 - \cos\omega_n(t-T)\right]$$

Si lo hicieramos por superposición



para $t < T$

$$x(t) = \frac{F_o r(t)}{T} = \frac{F_o}{KT}\left[t - \frac{\operatorname{sen}\omega_n t}{\omega_n}\right]$$



para $T < t$

$$x(t) = \frac{F_o}{T} r(t) - \frac{F_o}{T} r(t-T) - F_o\, g(t-T)$$

$$x(t) = \frac{F_o}{T}\,\frac{1}{K}\left[t - \frac{\operatorname{sen}\omega_n t}{\omega_n}\right] - \frac{F_o}{T}\,\frac{1}{k}\left[t - T - \frac{\operatorname{sen}\omega_n (t-T)}{\omega_n}\right] - \frac{F_o}{k}\left[1 - \cos\omega_n (t-T)\right]$$

$$\therefore \boxed{x(t) = \frac{F_o}{TK}\left[T + \frac{\operatorname{sen}\omega_n (t-T)}{\omega_n} - \frac{\operatorname{sen}\omega_n t}{\omega_n}\right] - \frac{F_o}{K}\left[1 - \cos\omega_n (t-T)\right]}$$